A generalization of the functional equation of distributivity

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§ 1. Introduction

1. The functional equation

(1) $$F[G(x, y), u] = G[F(x, u), F(y, u)]$$

of distributivity can be generalized in various ways. Such a generalization is

(2) $$F[G(x, y), u] = H[K(x, u), L(y, u)]$$

($x \in X, y \in Y, u \in U$; $G: X \times Y \to Z$; $F: Z \times U \to Z'$; $K: X \times U \to X'$; $L: Y \times U \to Y'$; $H: X' \times Y' \to Z'$), which is, at the same time, a generalization also of the following functional equations containing functions of two real variables:

$$E_k(x, y)u^k = E_k(xu, yu) \quad \text{(EULER's homogeneous function)},$$
$$F[F(x, y), u] = F[x, L(y, u)] \quad \text{(transformation)},$$
$$F[F(x, y), u] = F[x, F(y, u)] \quad \text{(associativity)},$$
$$F[F(x, y), u] = F(x, y + u) \quad \text{(translation)},$$
$$F(x, y) = F[F(x, u), F(y, u)] \quad \text{(transitivity)}.$$

Moreover, (2) is also in connection with the functional equation

$$B[B(x, y), B(u, v)] = B[B(x, u), B(y, v)]$$

of bisymmetry, namely, this last equation with $v = v_0$ (constant) means that $B(x, y)$ satisfies a functional equation of distributive type (2), where

$$K = H = G = B, \quad F(x, u) = B[x, B(u, v_0)], \quad L(x, u) = B(x, v_0),$$

and the solution thus obtained satisfies also certain further restrictions as the functional equation of bisymmetry with arbitrary v.

Another generalization can be obtained by considering complex numbers or multidimensional vectors instead of real variables and functions. Moreover, if the domain of definition is an arbitrary set, e. g. a quasigroup, then (2) can be interpreted as a system of homotopism resp. isotopism relations between two operations.[1]) If, e. g., $G=H=A(x, y)$ is a binary operation on 3-dimensional euclidean space X_3 and

$$F=K=L=R(x, u)$$

are rotations of this space, then we get the functional equation

$$R[A(x, y), u]=A[R(x, u), R(y, u)]$$

which means that $A(x, y)$ is a rotation-automorphic operation.

2. These generalizations and special distributive type functional equations play an important role in several investigations as in the theory of geometric objects [6—8], in the axiomatization of probability calculus [2] and vector operations [1], etc. Previously [4—8] (1) and (2) were treated under certain invertibility and differentiability conditions. In the present paper we shall see the solution of (2) where

(3) $$X=X'=Y=Y'=Z=Z'=T \text{ resp. } U$$

are domains in n- resp. m-dimensional spaces. If

$$X, X', Y, Y', Z, Z'$$

are different domains in X_n, then choosing suitable 1—to—1 mappings $h, \varkappa$ λ, ξ, η we consider the functions

$$h[F(z, u)], \quad \varkappa\{K[\xi(z), u]\}, \quad \lambda\{L[\eta(z), u]\} \qquad (Z\times U\to Z),$$
$$G[\xi(x), \eta(y)], \quad h\{H[\varkappa^{-1}(x), \lambda^{-1}(y)]\} \qquad (Z\times Z\to Z),$$

instead of F, K, L, G, H figuring in (2), for which a distributive type functional equation similar to (2) holds too. Therefore, in what follows, we shall consider only the case (3).

In § 2 the existence of an annullator $O=(0, 0, \ldots)\in U$ will be supposed for which

(4) $$K(x, O)=x_0, \; L(x, O)=y_0 \qquad \text{(constant)} \qquad (x\in T),$$

In § 3 the special case $n=1$ will be treated under weaker suppositions than in § 2 and using another method of solution. § 4 deals also with the case $n=1$, where $K=L=F$, but without supposing the existence of an annullator O. We shall conclude that the solutions G, H are isotopic to the addition resp. isomorphic to an E_1 function on reals, further, F, K, L are operator-isotopic to the affinities

$$x\to \mathbf{C}x+r, \quad \mathbf{C}=\mathbf{C}(u), \quad r=r(u).$$

[1]) See [6]. Here the isotopism resp. isomorphism etc. of a structure with operation xy will be called isotopism resp. isomorphism etc. of the operation xy. This will cause no misunderstanding.

§ 5 deals with the rotation- and homothety-automorphic operations $A(x, y)$ in X_3. We prove that such an A on a continuous abelian group is isomorphic to the addition and a suitable isomorphism is $x \to |x|^\gamma x$. Another example will be given: the rotation- and homothety-automorphic operations on X_n which are differentiable at $x = y = O$.

§ 2. T-algebras in X_n with h-operators and with annullator

1. A domain T will be called a *topological algebra* (briefly *T-algebra*) in X_n, if a differentiable binary operation $H(x, y)$ $(T \times T \to T)$ is defined on $T \subseteq X_n$ such that

$$|\partial_{x^i} H^j(x, y)|,\ |\partial_{y^i} H^j(x, y)| \neq 0 \qquad (i, j = 1, 2, \ldots, n)$$

hold. The elements $u \in U \subseteq X_m$ are called *homotopism operators* (briefly *h-operators*), if there exist mappings

$$F(x, u), \quad K(x, u), \quad L(x, u), \qquad (T \times U \to T)$$

such that (2), (3) hold. A h-operator is said to be an *isotopism operator* (briefly *i-operator*), if these mappings of T onto T are 1—to—1 hence invertible.[2]) An operator $O \in U$ is called *annullator*, if (4) holds. We state the following

Theorem 1. *Let T be a T-algebra in X_n with the h-operator set $U \subseteq X_m$ having an i-operator e and an annullator O such that*

$$\text{(I)} \qquad \begin{cases} f(x) = \partial_{u^i} F(x, O),\ \alpha(x) = \partial_{u^i} K(x, O),\ \beta(x) = \partial_{u^i} L(x, O) \quad (T \to X_n) \\ \text{are bounded and invertible mappings.} \end{cases}$$

Then T is topologically isotopic to the additive vector group in X_n having affinities as h-operators.

More exactly, then the most general solution of the functional equation (2) is

$$\text{(5)} \qquad \begin{cases} G(x, y) = f^{-1}[\varphi(x) + \psi(y)], \\ H(x, y) = \chi[\varkappa(x) + \lambda(y)], \\ F(x, u) = \chi[\mathbf{C}(u) f(x) + r(u) + s(u)], \\ K(x, u) = \varkappa^{-1}[\mathbf{C}(u)\varphi(x) + r(u)], \\ L(x, u) = \lambda^{-1}[\mathbf{C}(u)\psi(x) + s(u)], \end{cases}$$

where $f, \chi^{-1}, \varphi, \psi, \varkappa, \lambda$ $(T \to X_n)$ are arbitrary invertible mappings and r, s $(U \to X_n)$ are arbitrary mappings with the only restriction that the differen-

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2) A mapping $x (\in X) \to \varphi x$ $(\in X')$ is called 1—to—1, if the set X of the elements φx $(x \in X)$ is onefold. $x \to \varphi x$ is *onto* X', if $X' \subseteq \varphi X$. An *invertible* mapping is 1—to—1 and also onto.

tiability conditions shall be fulfilled, further, $\mathbf{C}(u)$ *is an arbitrary matrix with n rows and n columns for which*

(6) $$\mathbf{C}(O)=\mathbf{O}\,; \quad |\mathbf{C}(e)|,\ |\partial_u \mathbf{C}(O)| \neq 0$$

are satisfied.

Proof. Let us define the functions $\varphi, \psi, \chi, \varkappa, \lambda$ by

(7) $$\begin{cases} \varphi=\mathbf{A}\alpha, \ \psi=\mathbf{B}\beta, \ \mathbf{A}=\partial_x H(x_0, y_0), \ \mathbf{B}=\partial_y H(x_0, y_0), \\ \chi(x)=F[f^{-1}(x), e], \ \varkappa[K(x, e)]=\varphi(x), \ \lambda[L(x, e)]=\psi(x). \end{cases}$$

Differentiating (2) with respect to u^1 and putting $u=O$, we get (5_1):

$$f[G(x, y)]=\varphi(x)+\psi(y).$$

Substituting this into (2), with $u=e$ we have

$$H(\xi, \eta)=H[K(x, e), L(y, e)]=F[G(x, y), e]=F\{f^{-1}[\varphi(x)+\psi(y)], e\}=$$
$$=\chi\{\varkappa[K(x, e)]+\lambda[L(y, e)]\}=\chi[\varkappa(\xi)+\lambda(\eta)]$$

and this is (5_2).

Substituting again G and H into (2), we see

$$F\{f^{-1}[\varphi(x)+\psi(y)], u\}=\chi\{\varkappa[K(x, u)]+\lambda[L(y, u)]\}$$

from which, by keeping u fixed and denoting

(8) $$\begin{cases} \omega(x)=\chi^{-1}\{F[f^{-1}(x), u]\}, \\ \varrho(x)=\varkappa\{K[\varphi^{-1}(x), u]\}, \\ \sigma(x)=\lambda\{L[\psi^{-1}(x), u]\}, \end{cases}$$

the generalized CAUCHY functional equation

(9) $$\omega(x+y)=\varrho(x)+\eta(y) \qquad (x, y \in X_n;\ \varrho, \sigma, \omega : X_n \to X_n)$$

follows.

But here we have the

Lemma 1. *The most general bounded and one valued (not necessarily onto and 1—to—1) solutions of* (9) *are*

(10) $$\omega(x)=\mathbf{C}x+r+s, \quad \varrho(x)=\mathbf{C}x+r, \quad \sigma(x)=\mathbf{C}x+s,$$

where $\mathbf{C}$ *is an arbitrary matrix with n rows and columns and r, s are arbitrary n-dimensional vectors (depending on u).*

Taking (8) into account, (10) gives the solutions (5_3)—(5_5). Lemma 1 will be proved in § 2.

On the other hand, (5) really satisfies (2) and the initial conditions, if the restrictions (6) are fulfilled.

2. A similar theorem can be proved in the case where f, α, β map T onto the different domains $T_f, T_\alpha, T_\beta \subseteqq X_n$ respectively. Then the mappings

$$\varphi(T \to \mathbf{A}\, T_\alpha = T_\varphi), \quad \psi(T \to \mathbf{B}\, T_\beta = T_\psi), \quad \chi^{-1}(T \to T_f),$$
$$\varkappa(T \to T_\varphi), \quad \lambda(T \to T_\psi)$$

are 1--to—1 also in this case but not necessarily onto the whole of X_n and we must prove Lemma 1, where

$$\varrho(T_\varphi \to T_\varphi), \quad \sigma(T_\psi \to T_\psi), \quad \omega(T_f \to T_f = T_\varphi + T_\psi)$$

are bounded mappings but not necessarily onto. For this purpose we consider the arbitrarily fixed elements $x_0 \in T_\varphi$, $y_0 \in T_\psi$ and we define

$$\delta(x) = \omega(x + x_0 + y_0) - \omega(x_0 + y_0) =$$
$$= \varrho(x + x_0) - \varrho(x_0) = \sigma(x + y_0) - \sigma(y_0),$$

then we get

$$\delta(x+y) = \omega(x + y + x_0 + y_0) - \omega(x_0 + y_0) =$$
$$= \varrho(x + x_0) - \varrho(x_0) + \sigma(y + y_0) - \sigma(y_0) = \delta(x) + \delta(y).$$

Here x and y lay in a neighbourhood of O and so we get the most general bounded solution $\delta(x) = \mathbf{C}x$, i. e.,

$$\begin{cases} \varrho(x) = \mathbf{C}(x - x_0) + \varrho(x_0) = \mathbf{C}x + r, \\ \sigma(y) = \mathbf{C}(y - y_0) + \sigma(y_0) = \mathbf{C}y + s, \\ \omega(x+y) = \mathbf{C}(x + y - x_0 - y_0) + \omega(x_0 + y_0) = \mathbf{C}(x+y) + r + s \end{cases}$$

on a neighbourhood of x_0 resp. y_0.

Since x_0, y_0 are arbitrary, we have the solution (10) on a neighbourhood of every x_0, y_0, futhermore, being this solution unique in an *open* neighbourhood of any fixed x_0, y_0 we have the same solution on the union of these open subsets, i. e., on the whole domain of definition.

On the other hand, (10) obviously satisfies (9) with arbitrary $\mathbf{C}, r, s$.

3. In the special case $G = H$, $F = K = L$, under the suppositions of Theorem 1 we have the general solution

$$\begin{cases} G(x, y) = f^{-1}[\mathbf{A}f(x) + \mathbf{B}f(y)], \\ F(x, u) = f^{-1}[\mathbf{C}(u)f(x) + r(u)], \end{cases}$$

where the restrictions (6) and also

$$\mathbf{AC} = \mathbf{CA}, \mathbf{BC} = \mathbf{CB}; \ (\mathbf{A} + \mathbf{B})r = r$$

are fulfilled. This latter assertions can be proved immediately.

It might be observed that in Theorem 1 the differentiability of $H(x, y)$ was used only at $x = x_0$, $y = y_0$.

§ 3. T-algebras in X_1 with h-operators and with annullator

1. Let us consider the solution of (2) on

$$T = X = X' = Y = Y' = Z = Z' \subseteq X_1$$

and on U. Here we do not suppose U to be a domain in an euclidean space and also the differentiability conditions (I) can be replaced by weaker ones. We state

Theorem 2. *Let T be a T-algebra in X_1 with operation $H(x, y)$ and with a set U of h-operators having an i-operator e and an annullator O, further, having a metric subspace $U_0 \subseteq U$ such that*

(i) *U_0 and the set of its limit points contain O and e;*

(ii) *$K(x, u)$, $L(x, u)$ are continuous on U_0 at $u = O$ for every fixed $x \in T$;*

(iii) *$\partial_x F(x, u)$, $\partial_x K(x, u)$, $\partial_x L(x, u) \neq 0$ for $u \in U_0 - O$.*

Then T is (topologically) isotopic to the real additive group with affinities as h-operators. More exactly, then the solution of the functional equation (2) is

$$\left\{\begin{array}{l} G(x, y) = f^{-1}[\varphi(x) + \psi(y)], \\ H(x, y) = \chi[\varkappa(x) + \lambda(y)], \\ F(x, u) = \chi[f(x)c(u) + r(u) + s(u)], \\ K(x, u) = \varkappa^{-1}[\varphi(x)c(u) + r(u)], \\ L(x, u) = \lambda^{-1}[\psi(x)c(u) + s(u)], \end{array}\right. \tag{11}$$

where $c(u)$ and $r(u)$, $s(u)$ are on U_0 continuous but otherwise arbitrary functions with the only restriction

$$c(O) = 0 \notin c(U_0 - 0),$$

further, $f, \varphi, \psi, \chi, \varkappa, \lambda$ are arbitrary once differentiable functions with non zero derivatives.

Proof. Keeping $u \in U_0$, (2) shows that also $G(x, y)$ is differentiable. Therefore differentiating (2) with respect to x resp. y, we obtain

$$\left\{\begin{array}{l} F_1[G(x, y), u]\, G_1(x, y) = H_1[K(x, u), L(y, u)]\, K_1(x, u), \\ F_1[G(x, y), u]\, G_2(x, y) = H_2[K(x, u), L(y, u)]\, L_1(y, u), \end{array}\right. \tag{12}$$

where the indices 1,2 denote the partial derivatives with respect to the first resp. second variables. Keeping $u \in U_0 - O$, these equations and (iii) imply $G_1(x, y)$, $G_2(x, y) \neq 0$.

Let us define

$$\Phi(x, y, \xi, \eta) = \frac{\dfrac{G_1(x, y)}{G_2(x, y)} \dfrac{G_1(\xi, \eta)}{G_2(\xi, \eta)}}{\dfrac{G_1(x, \eta)}{G_2(x, \eta)} \dfrac{G_1(\xi, y)}{G_2(\xi, y)}}$$

and $\Psi(x, y, \xi, \eta)$ similarly with H's instead of G's, then, forming

$$\frac{G_1(x, y)}{G_2(x, y)} = \frac{H_1[K(x, u), L(y, u)]}{H_2[K(x, u), L(y, u)]} \frac{K_1(x, u)}{L_1(y, u)}$$

and taking also (i), (ii) into account, we see that

$$\Phi(x, y, \xi, \eta) = \Psi[K(x, u), L(y, u), K(\xi, u), L(\eta, u)] = \\ = \Psi(x_0, y_0, x_0, y_0) = 1.$$

So keeping ξ, η fixed,

$$\frac{G_1(x, y)}{G_2(x, y)} = \frac{G_2(\xi, \eta) \dfrac{G_1(x, \eta)}{G_2(x, \eta)}}{G_1(\xi, \eta) \dfrac{G_2(\xi, y)}{G_1(\xi, y)}} = \frac{\varphi'(x)}{\psi'(y)}$$

or, what is the same,

$$\begin{vmatrix} G_1(x, y) & G_2(x, y) \\ \varphi'(x) & \psi'(y) \end{vmatrix} = 0$$

involves the existence of a function f for which

$$f[G(x, y)] = \varphi(x) + \psi(y)$$

holds. Since the derivatives of G do not vanish,

$$\varphi(x) = G_1(\xi, \eta) \int \frac{G_1(x, \eta)}{G_2(x, \eta)} dx$$

is a strictly monotonic function with non zero derivative; $\psi(x)$ and consequently $f(x)$ too have the same property.

Thus we have (11_1). Now, we define $\chi, \varkappa, \lambda$ similarly as in (7) was done and the further proof goes in the same way as that of Theorem 1.

2. In the special case $G = H$, $K = L = F$ the supposition of the existence of an i-operator e can be omitted. Then we get the solution

$$\left\{ \begin{aligned} &F(x, u) = f^{-1}[f(x)g(u) + h(u)], \\ &Gx, y = f^{-1}[af(x) + bf(y) + c], \end{aligned} \right. \tag{13}$$

where we have the restrictions

$$c[g(u) - 1] \equiv (a + b - 1)h(u), \qquad g(O) = 0. \tag{14}$$

In fact, $G(x, y) = f^{-1}[\varphi(x) + \psi(y)]$ can be proved similarly as in Theorem 2. Putting this G into (2), we obtain

$$F\{f^{-1}[\varphi(x) + \psi(y)], u\} = f^{-1}\{\varphi[F(x, u)] + \psi[F(y, u)]\}$$

from which, keeping u constant and denoting

$$\omega(x) = f\{F[f^{-1}(x), u]\}, \quad \varrho(x) = \varphi\{F[\varphi^{-1}(x), u]\}, \quad \sigma(x) = \psi\{F[\psi^{-1}(x), u]\}$$

the generalized CAUCHY functional equation

$$\omega(x + y) = \varrho(x) + \sigma(y)$$

follows, the (differentiable) solution of which is

$$\omega(x)=gx+h, \quad \varrho(x)=gx+r, \quad \sigma(x)=gx+s \qquad (h=r+s).$$

Here g, h depend on u. This gives

$$F(x,u)=f^{-1}[f(x)g(u)+h(u)]=\varphi^{-1}[\varphi(x)g(u)+r(u)]=$$
$$=\psi^{-1}[\psi(x)g(u)+s(u)]$$

which proves (13_1) and shows that f, φ, ψ are not independent from each other, as, by denoting $\gamma(x)=\varphi[f^{-1}(x)]$, we must have

$$\gamma(gx+h)=g\gamma(x)+h.$$

By differentiating we get

$$\gamma'(x)=\gamma'(gx+h)=\gamma'\left(g^n x+\frac{1-g^n}{1-g}h\right)=\cdots=\gamma'\left(\frac{h}{1-g}\right)=a \qquad \text{(constant)}$$

as $|g(u)|\neq 1$ since $g(O)=0$ holds necessarily on account of (13_1). So we obtain

$$\lim g^n \to 0 \begin{cases} \text{for } n\to\infty, & \text{if } |g|<1, \\ \text{for } n\to-\infty, & \text{if } |g|>1. \end{cases}$$

Thus we have

$$\gamma(x)=\varphi[f^{-1}(x)]=ax+c_1, \quad \varphi(x)=af(x)+c_1,$$

and similarly also

$$\psi(x)=bf(x)+c_2$$

and (13_2) where $c=c_1+c_2$.

On the other hand, (13) really satisfies (1) if (14) is fulfilled.

3. It might be remarked that Theorem 2 is related to the following theorem: The most general measurable solution of the functional equation

$$f[G(x,y)]=f(x)+f(y)$$

is $f(x)=cf_0(x)$, where f_0 is an invertible bounded solution, supposed that such a solution exists. Here we have $F(x,e)=f_0(x)$, $F(x,O)=0$.

§ 4. Transitive system of T-endomorphisms on real T-quasigroups

1. An interval I will be called a *T-quasigroup* if an invertible operation $G(x,y)$ ($I\times I\to I$) is defined[3]) such that I forms a T-algebra in X_1. A mapping $x\to F(x,u)$ will be called a *homomorphism* on I, if

$$F[G(x,y),u]=H[F(x,u),F(y,u)] \qquad (x,y\in I;\ u\in U) \tag{15}$$

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[3]) A set Q in which a binary operation xy ($Q\times Q\to Q$) is defined is called a *quasigroup*, if both $x\to xy$ and $y\to xy$ are invertible.

holds. An invertible homomorphism is an *isomorphism*. In a similar sense we define the *endomorphism* and *automorphism* in the case, where $H=G$. A system of endomorphisms $x\to F(x,u)$ will be called *topological endomorphisms* (briefly *T-endomorphisms*), if $F(x,u)$ is a twice differentiable function of x and $\partial_x F(x,u)\neq 0$. Endomorphisms $x\to F(x,u)$ are called *transitive* for which $F(x_0,U)=I$ at least for one $x_0\in I$.

We state the following lemmas:

Lemma 2. *Let Q be a quasigroup with the operation xy. Every homomorphism of Q is also a homomorphism of the left and right quotient quasigroup of Q (and vice versa) in which the operation xy^{-1} resp. ${}^{-1}xy$ is defined by*

$$(xy^{-1})y=x,\quad x({}^{-1}xy)=y.$$

Lemma 3. *Let Q be a continuous quasigroup on reals with operation xy. Then at least one of the mappings*

$$x\to\mu x=xx,\ xx^{-1},\ {}^{-1}xx\qquad (Q\to Q)$$

is invertible.

Lemma 4. *The mappings $F_u x=F(x,u)$ $(I\times U\to I')$ form a system of homomorphisms of an algebra with isomorphism F_e, if and only if $N_v x=F_e^{-1}F_u x$ $(I\times U\to I)$ is a system of endomorphisms with identity $N_e x=x$.*

Lemma 5. *Let Q be an algebra (not necessarily a quasigroup) in which an operation xy is defined for which the mapping $\mu x=xx$ is invertible. Then Q^* with the operation $x*y=\mu^{-1}(xy)$ is an idempotent algebra $(x*x=x)$. A mapping $x\to N(x,u)$ is an endomorphism of Q, if and only if it is an endomorphism of Q^*.*

Lemma 6. *Let $G(x,y)$ $(I\times I\to I)$, $H(x,y)$ $(I'\times I'\to I')$ be T-quasigroup operations on reals and $F(x,u)$ a differentiable function of x. Suppose that the mappings $x\to F(x,u)$ are homomorphisms of I into I'. Then $u=u_0$ is an annullator, if and only if $\partial_x F(x,u_0)=0$.*

Proof of the Lemmas. The proof of the Lemmas 2—5 is not difficult [6]. In order to prove Lemma 6, let us consider

$$(12_1)\qquad F_1[G(x,y),u]G_1(x,y)=H_1[F(x,u)\,F(y,u)]F_1(x,u).$$

If $F_1(x,u_0)=0$ for a fixed u_0, then we see that $F_1(t,u_0)=0$ holds for every $t=G(x,y)$, hence $F(t,u_0)$ is independent from t, i. e., u_0 is an annullator.

2. On account of the above lemmas, in order to solve the functional equation (15) we can restrict ourselves without loss of generality to the functional equation

$$(16)\qquad \begin{cases} N[M(x,y),u]=M[N(x,u),N(y,u)],\\ M(x,x)=x,\ N(x,e)=x,\ N(x_0,U)=I \qquad (x,y\in I),\end{cases}$$

if the propositions of these lemmas are fulfilled. Since § 3 gives the solution in the case where an annullator exists, taking the Lemma 6 into account, we consider only T-endomorphisms, where $\partial_x F(x, e) \neq 0$.

We state the following

Theorem 3. *Let $N(x, u)$ $(I \times U \to I)$ be a transitive system of T-endomorphisms (with the identical operator $u = e$) working on an idempotent T-algebra $I \subseteq X_1$ (not necessarily a quasigroup) with the operation $M(x, y)$ $(I \times I \to I)$ having a continuous second order derivative $\partial_{xy} M(x, y)$ for $y = x$.*

Then I is (topologically) isomorphic to an I' with an operation $\Phi(x, y)$ (isomorphic to an E_1 homogeneous function[4]) having affinities as T-endomorphisms.

More exactly, the most general solution of the functional equation (16) *under the conditions*

1. $\partial_x M(x, y)$, $\partial_y M(x, y) \neq 0 \quad (x, y \in I)$;
2. *$N(x, u)$ is a twice differentiable function of x and $\partial_x N(x, u) \neq 0$;*
3. *$\partial_{xy} M(x, y)$ is continuous for $y = x$,*

is the following:

$$(17) \qquad \begin{cases} N(x, u) = f^{-1}[f(x) g(u) + h(u)], \\ M(x, y) = f^{-1}\{\varphi[f(x) - f(y)] + f(y)\}, \end{cases}$$

where f, φ are arbitrary twice differentiable functions with non zero derivative of the first order, and g, h are arbitrary functions with the restrictions

$$(18) \qquad \begin{cases} g \neq 0;\ g(e) = 1, h(e) = 0;\ f(I) = h(U);\ f(x_0) = 0; \\ \varphi(gx) = g\varphi(x),\ \varphi(0) = 0. \end{cases}$$

Proof. First we prove that

$$(19) \qquad M_1(x, x) M_2(x, x) = a \qquad \text{(constant)}.$$

To show this, let us differentiate (16_1) with respect to x resp. y. With $y = x = x_0$ we can form

$$\frac{M_1(x_0, x_0)}{M_2(x_0, x_0)} = \frac{M_1[N(x_0, u), N(x_0, u)]}{M_2[N(x_0, u), N(x_0, u)]} = \frac{M_1(x, x)}{M_2(x, x)} = \text{constant}.$$

On the other hand, differentiating (16_2) we obtain $M_1(x, x) + M_2(x, x) = 1$ which proves the statement.

Now, we reduce (16) to a functional-differential equation by differentiating it with respect to x and y:

$$N_{11}[M(x, y), u] M_1(x, y) M_2(x, y) + N_1[M(x, y), u] M_{12}(x, y) = \\ = M_{12}[N(x, u), N(y, u)] N_1(x, u) N_1(y, u).$$

[4]) See § 1.

If we put $y=x$, then by taking (16_2), (16_3) and (19) into account, we get

$$aN_{11}(x,u)+N_1(x,u)M_{12}(x,x)=M_{12}[N(x,u),N(x,u)]N_1(x,u)^2,$$

or, what is the same,

$$(-1/a)M_{12}[N(x,u),N(x,u)]N_1(x,u)+\frac{N_{11}(x,u)}{N_1(x,u)}=(-1/a)M_{12}(x,x).$$

By integrating we obtain

$$(-1/a)\int_{x_0}^{N(x,u)}M_{12}(s,s)ds+\log N_1(x,u)=(-1/a)\int_{x_0}^{x}M_{12}(s,s)ds+\log g(u),$$

i. e.,

$$N_1(x,u)\exp\Big[(-1/a)\int_{x_0}^{N(x,u)}M_{12}(s,s)ds\Big]=g(u)\exp\Big[(-1/a)\int_{x_0}^{x}M_1(s,s)ds\Big].$$

Integrating once again and introducing the function

$$f(x)=\int_{x_0}^{x}\exp\Big[(-1/a)\int_{x_0}^{t}M_{12}(s,s)ds\Big]dt$$

(having non zero derivative) we get $f[N(x,u)]=f(x)g(u)+h(u)$ which is (17_1).

In order to prove (17_2), we substitute (17_1) into (16_1):

$$f^{-1}\{f[M(x,y)]g(u)+h(u)\}=$$
$$=M\{f^{-1}[f(x)g(u)+h(u)],f^{-1}[f(y)g(u)+h(u)]\}$$

from which by denoting

$$\Phi(x,y)=f\{M[f^{-1}(x),f^{-1}(y)]\}$$

it follows the functional equation

(20) $$\Phi(x,y)g(u)+h(u)=\Phi[xg(u)+h(u),yg(u)+h(u)],$$

equivalent to (16_1), (17_1).

Three cases are possible: 1. $g(u)\equiv 1$; 2. $|g(u)|\equiv 1$ and $g(u_0)=-1$ at least for one u_0; 3. $|g(u)|\not\equiv 1$.

In the case $g(u)\equiv 1$, according to (16_4), u can be choosen such that $h(u)=-y$ and hence

(21) $$\Phi(x,y)=\varphi(x-y)+y,\quad \varphi(t)=\Phi(t,0)$$

hold. If $g(u_0)=-1$ and $|g(u)|\equiv 1$, then similarly we have the same solution but here φ has to satisfy $\varphi(-t)=-\varphi(t)$ as (20) must be satisfied. If $|g|\not\equiv 1$, then by differentiating we get

$$\left.\begin{aligned}\Phi_1(x,y)&=\Phi_1(gx+h,gy+h)=\cdots=c_1\\ \Phi_2(x,y)&=\Phi_2(gx+h,gy+h)=\cdots=c_2\end{aligned}\right\}\text{(constants)}$$

and, consequently, by integrating $\Phi(x,y)=px+qy+r$, where on account of (16_2) $p+q=1$, $r=0$.

The solution is also in this case of the form (21), but here $\varphi(t)=pt$ The solutions can be united in (21) where φ satisfies the restriction

$$\varphi(gx)=g\varphi(x),\ \varphi(0)=0.$$

The other restrictions (18) can be seen immediately.

Φ is obviously isomorphic to a function

$$E_1(x,y)=\varphi(x\,y)y$$

which is homogeneous.

Thus Theorem 3 is proved.

3. If we have the solutions M, N of the form (17), then we obtain the general form of F, G, H on account of Lemmas 4—5:

$$\begin{aligned}&F(x,u)=F_c N_u x=\nu[N(x,u)],\\&G(x,y)=\mu[M(x,y)],\\&H(x,y)=\nu\{G[\nu^{-1}(x),\nu^{-1}(y)]\}=\mu(\nu\{M[\nu^{-1}(x),\nu^{-1}(y)]\}),\end{aligned}$$

which with arbitrary functions ν, μ really satisfy (15), if $\mu[N(x,u)]=N[\mu(x),u]$ holds. This last equation can be written as

$$\mu\{f^{-1}[f(x)g(u)+h(u)]\}=f^{-1}\{f[\mu(x)]g(u)+h(u)\}$$

and this can be reduced to

$$\vartheta(gx+h)=g\vartheta(x)+h$$

by the notation $\vartheta=f\mu f^{-1}$.

4. If, e. g., U is a metric space and $x\to N(x,u)$ is a continuous, not identical constant mapping, then, on account of $N(x,e)=x$, I is the union of the transitivity sets $I_i=N(x_i,U)$.

Then the supposition $N(x_0,U)=I$, which was used only to prove

$$\frac{M_1(x,x)}{M_2(x,x)}=\frac{M_1[N(x_0,u),N(x_0,u)]}{M_2[N(x_0,u),N(x_0,u)]}=\frac{M_1(x_0,x_0)}{M_2(x_0,x_0)}\ \text{(constant)},$$

can be omitted.

§ 5. Examples

1. Let us consider the rotation automorphic vector operations $A(x,y)$ on a continuous abelian group in X_3. G. DARBOUX [3] has proved that such an $A(x,y)$ is topologically isomorphic to the addition, i. e., it has the form

$$f[A(x,y)]=f(x)+f(y),$$

where

$$f(x)=\varphi\cdot x^0=\varphi\cdot x/|x|$$

and φ is a topological function of $|x|$. Clearly,

$$y = f(x) = \varphi \cdot x^0 = |y| x^0$$

has the inverse function

$$x = f^{-1}(y) = \varphi^{-1}(|y|) x^0.$$

Let us examine the question what kind of operations $A(x, y)$ of this form are homothety-automorphic (homogeneous)! Homogeneity means that

$$\lambda A(x, y) = A(\lambda x, \lambda y)$$

or, in another form,

$$f[\lambda f^{-1}(x+y)] = f[\lambda f^{-1}(x)] + f[\lambda f^{-1}(y)].$$

Choosing $x = \xi x^0$, $y = \eta x^0$, further, by denoting

$$f[\lambda f^{-1}(x)] = f[\lambda \varphi^{-1}(\xi) x^0] = \varphi[\lambda \varphi^{-1}(\xi)] x^0 = \theta(\xi, \lambda) x^0,$$

we get the functional equation

$$\theta(\xi + \eta, \lambda) = \theta(\xi, \lambda) + \theta(\eta, \lambda)$$

the continuous solution of which is

$$\theta(\xi, \lambda) = \xi \vartheta(\lambda).$$

So we have

$$\varphi[\lambda \varphi^{-1}(\xi)] = \xi \vartheta(\lambda), \quad \varphi(\lambda \xi) = \vartheta(\lambda) \varphi(\xi)$$

which gives the solution

$$\varphi(\xi) = \xi^{\nu+1} \varphi(1)$$

by reducing it once again to CAUCHY's functional equation.

Finally, we have the solution

$$A(x, y) = f^{-1}[f(x) + f(y)],$$

where

$$f(x) = \varphi(|x|) x^0 = |x|^{\nu+1} x^0 = |x|^{\nu} x.$$

Here we can choose $\varphi(1) = 1$ without loss of generality.

Clearly, every $A(x, y)$ of this form is rotation-automorphic and homogeneous.

2. Let us consider the homogeneous rotation-automorphic $A(x, y)$ vector operations in X_n which are differentiable at $x = y = O$. Homogeneity means that

$$\lambda A(x, y) = A(\lambda x, \lambda y)$$

holds for every real λ value. If we differentiate this equation with respect to λ and substitute $\lambda = 0$ we obtain

$$A(x, y) = \mathbf{A} x + \mathbf{B} y.$$

Rotation automorphism means that $A(x, x)$ and $A(x, -x)$ do not depend from the rotations around x, that is

$$A(x, x) = (\mathbf{A} + \mathbf{B})x = \alpha x,$$
$$A(x, -x) = (\mathbf{A} - \mathbf{B})x = \beta x.$$

These involve that $\mathbf{A} + \mathbf{B} = \alpha\mathbf{E}$, $\mathbf{A} - \mathbf{B} = \beta\mathbf{E}$ where $\mathbf{E}$ is the unit matrix. Thus we have

$$\mathbf{A} = \frac{\alpha+\beta}{2}\mathbf{E} = \mu\mathbf{E}, \quad \mathbf{B} = \frac{\alpha-\beta}{2}\mathbf{E} = \nu\mathbf{E},$$

and the general form of $A(x, y)$ is

$$A(x, y) = \mu x + \nu y.$$

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